Corrugated surfaces with slow modulation and quasiclassical Weierstrass representation

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Abstract

Quasiclassical generalized Weierstrass representation for highly corrugated surfaces in $\mathbb{R}^3$ with slow modulation is proposed. Integrable deformations of such surfaces are described by the dispersionless modified Veselov-Novikov hierarchy.

1 Introduction

Surfaces, interfaces, fronts and their dynamics are the key ingredients in a number of very interesting phenomena from hydrodynamics, growth of crystals, theory of membranes to the string theory and gravity (see *e.g.* [1-5]). The most of the papers on this subject and results obtained have been concerned to a smooth case. On the other hand, irregular, corrugated surfaces also have attracted interest in various fields from the applied physics, technology to the pure mathematics (see *e.g* [6-13]).

In the present paper we propose a Weierstrass type representation for highly corrugated surfaces with a slow modulation in the three dimensional Euclidean space $\mathbb{R}^3$. It is the quasiclassical limit of the generalized Weierstrass representation (GWR) for surfaces in $\mathbb{R}^3$ introduced in [14,15]. This GWR was based on the two-dimensional Dirac equation and it allows to construct any surface in $\mathbb{R}^3$. The hierarchy of the modified Veselov-Novikov (mVN) equations provides us with the integrable deformations of surfaces [14,15].

The quasiclassical GWR is based on the quasiclassical limit of the Dirac equation. It allows us to construct surfaces in $\mathbb{R}^3$ with highly oscillating (corrugated) profiles and slow modulations of these oscillations characterized by a small parameter $\varepsilon = \frac{l}{L}$ where l and L are typical scales of oscillations and modulations, respectively. In the lowest order in ε the coordinates X^j $(j = 1, 2, 3)$

of such surfaces are of the form

$$
\begin{aligned}
X^1 + iX^2 &= A(\varepsilon z, \varepsilon\overline{z}) \exp\left[-2i\frac{S(\varepsilon z, \varepsilon\overline{z})}{\varepsilon}\right], \\
X^3 &= \frac{1}{\varepsilon} B(\varepsilon z, \varepsilon\overline{z})
\end{aligned}
\tag{1.1}
$$

where z and $\overline{z}$ are the conformal coordinates on a surface, A and B some smooth functions and S is a solution of the eikonal equation. The corresponding metric and mean curvature are finite functions of the slow variables $\varepsilon z, \varepsilon\overline{z}$ while the Gaussian curvature is of the order ε^2.

Integrable deformations of such corrugated surfaces are induced by the hierarchy of dispersionless mVN equations . These deformations preserve the quasiclassical Willmore functional (Canham-Helfrich bending energy for membranes or the Polyakov extrinsic action for strings (see *e.g.* [3,4]). The quasiclassical limit of the Gauss map and Kenmotsu representation for surfaces in $\mathbb{R}^3$ of above type are discussed too.

2 Generalized Weierstrass representation for surfaces in $\mathbb{R}^3$

The generalized Weierstrass representation (GWR) proposed in [14.15] is based on the linear system (two-dimensional Dirac equation)

$$
\begin{aligned}
\Psi_z &= p\Phi, \\
\Phi_{\overline{z}} &= -p\Psi
\end{aligned}
\tag{2.1}
$$

where Ψ and Φ are complex -valued functions of $z, \overline{z} \in \mathbb{C}$ (bar denotes a complex conjugation) and $p(z, \overline{z})$ is a real-valued function. One defines three real-valued functions $X^j(z, \overline{z}), j = 1, 2, 3$ by the formulae

$$
\begin{aligned}
X^1 + iX^2 &= i\int_\Gamma (\overline{\Psi}^2 dz\prime - \overline{\Phi}^2 d\overline{z\prime}), \\
X^1 - iX^2 &= i\int_\Gamma (\Phi^2 dz\prime - \Psi^2 d\overline{z\prime}), \\
X^3 &= -\int_\Gamma (\overline{\Psi}\Phi dz\prime + \Psi\overline{\Phi} d\overline{z\prime})
\end{aligned}
\tag{2.2}
$$

where Γ is an arbitrary contour in $\mathbb{C}$.

Theorem 1 [14,15]. *For any function $p(z, \overline{z})$ and any solution (Ψ, Φ) of the system (2.1) the formulae (2.2) define a conformal immersion of a surface into $\mathbb{R}^3$ with the induced metric*

$$
ds^2 = u^2 dz d\overline{z}, \tag{2.3}
$$

the Gaussian curvature

$$
K = -\frac{4}{u^2}(\log u)_{z\overline{z}}, \tag{2.4}
$$

mean curvature

$$H = 2\frac{p}{u}, \tag{2.5}$$

and the Willmore functional given by

$$W \stackrel{def}{=} \iint_G W^2[ds] = 4\iint_G p^2 dxdy, \tag{2.6}$$

where $u = |\Psi|^2 + |\Phi|^2$ *and* $z = x + iy$.

Morever, any regular surface in $\mathbb{R}^3$ can be constructed via the GWR (2.1),(2.2).

Integrable dynamics of surfaces constructed via the GWR (2.2) is induced by the integrable evolutions of the potential $p(z, \overline{z}, t)$ and the functions Ψ, Φ with respect the deformation parameter t. They are given by the modified Veselov-Novikov (mVN) hierarchy [14,15]. The simplest example is the mVN equation

$$\begin{aligned} p_t + p_{zzz} + p_{\overline{zzz}} + 3\omega p_z + 3\overline{\omega} p_{\overline{z}} + \frac{3}{2}p\omega_z + \frac{3}{2}p\overline{\omega}_{\overline{z}} &= 0, \\ \omega_{\overline{z}} &= (p^2)_z \end{aligned} \tag{2.7}$$

while Ψ and Φ obey the system of linear equations

$$\begin{aligned} \Psi_t + \Psi_{zzz} + \Psi_{\overline{zzz}} - 3p_z\Phi_z + 3\overline{\omega}\Psi_{\overline{z}} + \frac{3}{2}\overline{\omega}_{\overline{z}}\Psi + 3p\omega\Phi &= 0, \\ \Phi_t + \Phi_{zzz} + \Phi_{\overline{zzz}} + 3\omega\Phi_z + 3p_{\overline{z}}\Psi_{\overline{z}} - 3p\overline{\omega}\Psi + \frac{3}{2}\omega_z\Phi &= 0. \end{aligned} \tag{2.8}$$

The mVN equation (2.7) and the whole mVN hierarchy are amenable to the inverse spectral transform method [16,15] and they have a number of remarkable properties typical for integrable 2+1-dimensional equations. Integrable dynamics of surfaces in $\mathbb{R}^3$ inherits all these properties [15]. One of the remarkable features of such dynamics is that the Willmore functional W (2.6) remains invariant ($W_t = 0$) [17,18]. In virtue of the linearity of the basic problem (2.1) the GWR is quite a useful tool to study various problems in physics and mathematics (see *e.g.* [18-21]).

A different representation of surfaces in $\mathbb{R}^3$ has been proposed in [22]. It is based on the following parametrisation of the Gauss map

$$\overrightarrow{G} = (1 - f^2, i(1 + f^2), 2f), \tag{2.9}$$

where $f(z, \overline{z})$ is a complex-valued function. Then the Kenmotsu representation of a surface is given by [22]

$$\overrightarrow{X}(z, \overline{z}) = \mathrm{Re}\left(\int_\Gamma \eta \overrightarrow{G} dz\prime\right), \tag{2.10}$$

where the function η obeys the compatibility condition

$$(\log \eta)_{\overline{z}} = -\frac{2\overline{f} f_{\overline{z}}}{1 + |f|^2}, \tag{2.11}$$

The Kenmotsu representation (2.9)-(2.11) and the GWR (2.1), (2.2) are equivalent to each other. The relation between the functions (f, η) and (Ψ, Φ) is the following [17,18]

$$f = i\frac{\overline{\Psi}}{\Phi}, \eta = i\Phi^2, \tag{2.12}$$

while

$$p = -\frac{\eta f_{\overline{z}}}{|\eta| \left(1 + |f|^2\right)}.$$

Both the GWR and Kenmotsu representations have been widely used to study properties of generic surfaces and special classes of surfaces, in particular, of the constant mean curvature surfaces.

3 Quasiclassical Weierstrass representation.

In this paper we shall consider a class of surfaces in $\mathbb{R}^3$ which can be characterized by two scales l and L such that the parameter $\varepsilon = \frac{l}{L} \ll 1$. A simple example of such a surface is provided by the profile of a slowly modulated wavetrain for which l is a typical wavelength and L is a typical length of modulation. Theory of such highly oscillating waves with slow modulations is well developed (see *e.g.* [1,23]). Borrowing the ideas of this Whitham (or nonlinear WKB) theory we study surfaces in $\mathbb{R}^3$ for which the coordinates X^1, X^2, X^3 have the form

$$X^i(z, \overline{z}) = \sum_{n=0}^{\infty} \varepsilon^n F_n^i \left(\frac{\overrightarrow{S}(\varepsilon z, \varepsilon \overline{z})}{\varepsilon}, \varepsilon z, \varepsilon \overline{z} \right), \quad i = 1, 2, 3 \tag{3.1}$$

where $\overrightarrow{S} = (S^1, S^2, S^3)$ and F_n^i are smooth functions of slow variables $\xi = \varepsilon z, \overline{\xi} = \varepsilon \overline{z}$ and the small parameter ε is defined above. The arguments $\frac{S^i}{\varepsilon}$ in F_n^i describe a relatively fast variation of a surface while the rest of arguments correspond to slow modulations.

There are different ways to specify functions F_n^i. Here we will consider one of them induced by the similar quasiclassical (WKB) limit of the GWR (2.1), (2.2).

Thus, we begin with the quasiclassical limit of the Dirac equation (2.1). Having in mind the discussion given above, we take

$$p = \sum_{n=0}^{\infty} \varepsilon^n p_n(\varepsilon z, \varepsilon \overline{z}),$$

$$\Psi = \exp\left(\frac{iS(\varepsilon z, \varepsilon \overline{z})}{\varepsilon}\right) \sum_{n=0}^{\infty} \varepsilon^n \Psi_n(\varepsilon z, \varepsilon \overline{z}), \tag{3.2}$$

$$\Phi = \exp\left(\frac{iS(\varepsilon z, \varepsilon \overline{z})}{\varepsilon}\right) \sum_{n=0}^{\infty} \varepsilon^n \Phi_n(\varepsilon z, \varepsilon \overline{z})$$

where S, Ψ_n, Φ_n are smooth functions of slow variables $\xi = \varepsilon z, \overline{\xi} = \varepsilon \overline{z}$ and $\overline{S} = S$. Substituting (3.2) into (2.1), one in zero order in ε gets

$$\begin{aligned} iS_\xi \Psi_0 - p_0 \Phi_0 &= 0, \\ p_0 \Psi_0 + iS_{\overline{\xi}} \Phi_0 &= 0 \end{aligned} \tag{3.3}$$

while in the order ε one has

$$\begin{aligned} iS_\xi \Psi_1 - p_0 \Phi_1 &= -\Psi_{0\xi} + p_1 \Phi_{0,} \\ p_0 \Psi_1 + iS_{\overline{\xi}} \Phi_1 &= -\Phi_{0\overline{\xi}} - p_1 \Psi_0. \end{aligned} \tag{3.4}$$

The existence of nontrivial solutions for the system (3.3) implies that S should obey the equation

$$\det \begin{vmatrix} iS_\xi & -p_0 \\ p_0 & iS_{\overline{\xi}} \end{vmatrix} = -S_\xi S_{\overline{\xi}} + p_0^2 = 0, \tag{3.5}$$

or ($\xi = \xi_1 + i\xi_2$)

$$S_{\xi_1}^2 + S_{\xi_2}^2 = 4p_0^2, \tag{3.6}$$

i.e. the Hamilton-Jacobi equation for the two-dimensional classical system with the potential $4p_0^2$ or the eikonal equation with the refraction index $4p_0^2$.

From equations (3.3) and (3.5) it follows that $|S_\xi| = \left|S_{\overline{\xi}}\right| = p_0$ and $|\Psi_0| = |\Phi_0|$. Equations (3.4) imply the $iS_{\overline{\xi}}(\Psi_{0\xi} - p_1\Phi_0) = p_0(\Phi_{0\overline{\xi}} + p_1\Psi_0)$.

Using the differential form of the formulae (2.2), *i.e.*

$$\left(X^1 + iX^2\right)_z = i\overline{\Psi}^2, \left(X^1 + iX^2\right)_{\overline{z}} = -i\overline{\Phi}^2, X_z^3 = -\overline{\Psi}\Phi, \tag{3.7}$$

one concludes that the coordinates X^i have the form

$$\begin{aligned} X^1 + iX^2 &= \sum_{n=0}^{\infty} \varepsilon^n A_n(\varepsilon z, \varepsilon \overline{z}) \exp\left(-\frac{2iS(\varepsilon z, \varepsilon \overline{z})}{\varepsilon}\right), \\ X^3 &= \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n B_n(\varepsilon z, \varepsilon \overline{z}) \end{aligned} \tag{3.8}$$

where A_n and B_n are smooth functions. In the lowest order in ε one has

$$A_0 = -\frac{\overline{\Psi}_0^2}{2S_\xi} = \frac{\overline{\Phi}_0^2}{2S_{\overline{\xi}}}, B_{0\xi} = -\overline{\Psi}_0 \Phi_0. \tag{3.9}$$

Note that both the expressions for A_0 are equivalent to each other due to equations (3.3) and (3.5).

Thus, we have the

Theorem 2 . *The quasiclassical GWR provides us with the highly corrugated (oscillating) surfaces with slow modulations for which the coordinates have the form (3.8) where the function S is a solution of the eikonal equation (3.5) and in the lowest order the functions A_0 and B_0 are given by (3.9),(3.3). In the limit $\varepsilon \to 0$ one has the following principal contributions to the metric*

$$ds^2 = 4\left|\Psi_0(\varepsilon z, \varepsilon \overline{z})\right|^4 dz d\overline{z} = \frac{4}{\varepsilon^2}\left|\Psi(\xi, \overline{\xi})\right|^4 d\xi d\overline{\xi}, \tag{3.10}$$

mean curvature

$$H_0(\xi, \overline{\xi}) = \frac{p_0(\xi, \overline{\xi})}{2\left|\Psi_0(\xi, \overline{\xi}\right|^2}, \tag{3.11}$$

the Gaussian curvature

$$K_0 = -\varepsilon^2 \frac{2}{\left|\Psi_0(\xi, \overline{\xi}\right|^2} \left(\log\left|\Psi_0\right|\right)_{\xi\overline{\xi}}, \tag{3.12}$$

and the Willmore functional

$$W_0 = 4\iint_G p_0^2(\varepsilon z, \varepsilon \overline{z}) dx dy = \frac{4}{\varepsilon^2}\iint_{G_\varepsilon} p_0^2(\xi, \overline{\xi}) d\xi_1 d\xi_2, \tag{3.13}$$

where G_ε is the rescaled domain $G(x = \frac{\xi_1}{\varepsilon}, y = \frac{\xi_2}{\varepsilon})$.

One can refer to surfaces in $\mathbb{R}^3$ given by the formulae (3.8)-(3.13) as the quasiclassical surfaces. They represent a subclass of surfaces of the type (3.1).

We note that, in virtue of (3.6), the Willmore functional W_0 is just the Dirichlet integral

$$W_0 = \frac{1}{\varepsilon^2}\iint_{G_\varepsilon}\left(S_{\xi_1}^2 + S_{\xi_2}^2\right) d\xi_1 d\xi_2. \tag{3.14}$$

So, due to the Dirichlet principle (see *e.g.* [24]) the problem of minimization of W_0 is equivalent to the Dirichlet boundary problem for the harmonic function in the domain G_ε. For surfaces with all $p_n = 0, n = 1, 2, 3...$. the formulae (3.13) and (3.14) give us not just the asymptotic expressions for the Willmore functional at $\varepsilon \to 0$, but the exact one.

The quasiclassical analogs of surfaces of constant mean curvature and surfaces with $H\sqrt{\det g} = 1$ (see *e.g.* [18]) correspond to the constraints $p_0 = 2\left|\Psi_0\right|^2$ and $4p_0\left|\Psi_0\right|^2 = 1$, respectively. In the very particular case $p_1 = 0$ and $\Psi_1 = \Phi_1 = 0$ one has $\Psi_{0\xi} = 0, \Phi_{0\overline{\xi}} = 0$, *i.e.* $\Psi_0 = \Psi_0(\overline{\xi}), \Phi_0 = \Phi_0(\xi)$. Consequently, the formulae (3.3), (3.8)-(3.13) generate developable surfaces ($K_0 = 0$) with the metric $ds_0^2 = \frac{4}{\varepsilon^2}\Psi_0^2(\overline{\xi})\overline{\Psi}_0^2(\xi) d\xi d\overline{\xi}$ which after the reparametization $d\xi \to dw = 2\overline{\Psi}_0^2(\xi) d\xi$ becomes $ds_0^2 = \frac{1}{\varepsilon^2} dw d\overline{w}$.

Now let us consider the quasiclassical limit of the Kenmotsu representation. In virtue of (2.12) one has

$$
\begin{aligned}
f &= \widetilde{f}\exp\left(-\frac{2iS(\varepsilon z,\varepsilon\overline{z})}{\varepsilon}\right),\\
\eta &= \widetilde{\eta}\exp\left(\frac{2iS(\varepsilon z,\varepsilon\overline{z})}{\varepsilon}\right)
\end{aligned}
\tag{3.15}
$$

where

$$
\begin{aligned}
\widetilde{f} &= \sum_{n=0}\varepsilon^n f_n(\varepsilon z,\varepsilon\overline{z}),\\
\widetilde{\eta} &= \sum_{n=0}\varepsilon^n \eta_n(\varepsilon z,\varepsilon\overline{z})
\end{aligned}
$$

and $f_0 = i\frac{\overline{\Psi_0}}{\Phi_0}, h_0 = i\Phi_0^2$ and so on. Using (3.15), one gets the quasiclassical Gauss map

$$
\overrightarrow{G_q} = \left(1-\widetilde{f}^2 e^{-\frac{4iS}{\varepsilon}}, i(1+\widetilde{f}^2 e^{-\frac{4iS}{\varepsilon}}), 2\widetilde{f}e^{-\frac{2iS}{\varepsilon}}\right)
$$

and, finally, the quasiclassical Kenmotsu representation

$$
\overrightarrow{X} = \mathrm{Re}\left\{\frac{1}{\varepsilon}\int_{\Gamma_\varepsilon} d\xi\prime\widetilde{f}(\xi\prime,\overline{\xi}\prime)\widetilde{\eta}(\xi\prime,\overline{\xi}\prime)\left(\widetilde{f}^{-1}e^{\frac{2iS}{\varepsilon}} - \widetilde{f}e^{-\frac{2iS}{\varepsilon}}, i(\widetilde{f}^{-1}e^{\frac{2iS}{\varepsilon}} + \widetilde{f}e^{-\frac{2iS}{\varepsilon}}), 2\right)\right\},
\tag{3.16}
$$

which , of course, is equivalent to the quasiclassical GWR (3.8).

Quasiclasical versions of the GWR's for surfaces in the 4-dimensional [25,26,21] and higher dimensional spaces can be constructed in a similar manner.

4 Integrable deformations via the dmVN hierarchy.

Deformations of quasiclassical surfaces described above are given by the dispersionless limit of the mVN hierarchy. To get this limit one, as usual (at the 1+1-dimensional case, see *e.g.* [27]), assumes that the dependence of all quantities on t is a slow one, *i.e.* $p = p(\varepsilon z, \varepsilon\overline{z}, \varepsilon t), S = S(\varepsilon z, \varepsilon\overline{z}, \varepsilon t)$ and so on. At the first and second orders in ε equation (2.7) gives ($\tau = \varepsilon t, \omega = \sum_{n=0}\varepsilon^n\omega_n(\varepsilon z, \varepsilon\overline{z}, \varepsilon t)$)

$$
\begin{aligned}
p_{0\tau} + 3\omega_0 p_{0\xi} + 3\overline{\omega_0}p_{0\overline{\xi}} + \frac{3}{2}p_0\omega_{0\xi} + \frac{3}{2}p_0\overline{\omega}_{0\overline{\xi}} &= 0,\\
\omega_{0\overline{\xi}} &= (p_0^2)_\xi
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
p_{1\tau} + 3\omega_1 p_{0\xi} + 3\omega_0 p_{1\xi} + 3\overline{\omega}_0 p_{1\overline{\xi}} + 3\overline{\omega}_1 p_{0\overline{\xi}} + &\\
+\frac{3}{2}p_1(\omega_{0\xi} + \overline{\omega}_{0\overline{\xi}}) + \frac{3}{2}p_0(\omega_{1\xi} + \overline{\omega}_{1\overline{\xi}}) &= 0,\\
\omega_{1\overline{\xi}} &= 2(p_0p_1)_\xi.
\end{aligned}
\tag{4.2}
$$

In the lowest order equation (2.8) is

$$M_0 \begin{pmatrix} \Psi_0 \\ \Phi_0 \end{pmatrix} = 0, \tag{4.3}$$

where

$$M_0 = \begin{pmatrix} i(S_\tau - S_\xi^3 - S_{\overline{\xi}}^3 + 3\overline{\omega}_0 S_{\overline{\xi}}), & 3p_0\omega_0 \\ -3p_0\overline{\omega}_0, & i(S_\tau - S_\xi^3 - S_{\overline{\xi}}^3 + 3\omega_0 S_\xi) \end{pmatrix} \tag{4.4}$$

With the use of (3.5) the condition $\det M_0 = 0$ assumes the form

$$\left(S_\tau - S_\xi^3 - S_{\overline{\xi}}^3\right)\left(S_\tau - S_\xi^3 - S_{\overline{\xi}}^3 + 3\omega_0 S_\xi + 3\overline{\omega}_0 S_{\overline{\xi}}\right) = 0, \tag{4.5}$$

Equation (4.1) is the dispersionless limit of the mVN equation (dmVN equation). It is equivalent to the compatibility condition of the linear systems (3.3) and (4.3). In a similar manner one constructs the whole dmVN hierarchy.

This hierarchy generates the integrable deformations of the quasiclassical surfaces described in the previous section via the τ- dependence of $p_0, p_{1,} \Psi_0, \Phi_0$ ect given by equations (4.1)-(4.5) and so on. These integrable deformations are very special from the geometrical viewpoint. Indeed, the dmVN equation (4.1) implies that

$$\left(p_o^2\right)_\tau + 3\left(\omega_0 p_0^2\right)_\xi + 3\left(\overline{\omega}_0 p_0^2\right)_{\overline{\xi}} = 0. \tag{4.6}$$

So, for periodic or rapidly decaying at $|\xi| \to \infty$ functions p_0 and ω_0 one has

$$W_{0\tau} = \frac{4}{\varepsilon^2} \iint_{G_\varepsilon} \left(p_0^2\right)_\tau d\xi_1 d\xi_2 = 0. \tag{4.7}$$

One can show that the Willmore functional W_0 (or Dirichlet integral (3.14)) is invariant under the whole dmVN hierarchy of deformations as well. One may suggest that the quasiclassical limit of the higher mVN integrals , discussed in [28], will provide us with the higher geometrical invariants for quasiclassical surfaces.

The formula (4.6) indicates also an interesting connection of the dmVN equation with the other known dispersionless equation. Indeed, denoting $p_0^2 = u$, one has

$$\begin{aligned} u_\tau + 3\left(\omega_0 u\right)_\xi + 3\left(\overline{\omega}_0 u\right)_{\overline{\xi}} &= 0, \\ \omega_{0\overline{\xi}} &= u_\xi. \end{aligned} \tag{4.8}$$

It is the dispersionless VN equation introduced in [29,30]. The dVN equation is equivalent to the compatibility condition for the two Hamilton-Jacobi equations

$$\begin{aligned} S_\xi S_{\overline{\xi}} - u &= 0, \\ S_\tau - S_\xi^3 - S_{\overline{\xi}}^3 + 3\omega_0 S_\xi + 3\overline{\omega}_0 S_{\overline{\xi}} &= 0. \end{aligned} \tag{4.9}$$

These equations show that the whole theory of the dVN hierarchy can be developed without any reference to its dispersive version. Within the quasiclassical $\overline{\partial}-$ dressing method the dVN hierarchy has been studied in [31] in connection with the problems of nonlinear geometrical optics in the so-called Cole-Cole media. In order to apply the results obtained for the dVN hierarchy to the dmVN hierarchy one, due to the relation $u = p_0^2$, should be able to select effectively the positive solutions of the dVN hierarchy. This problem and also the application of the $\overline{\partial}-$ dressing method directly to the dmVN hierarchy will be considered elsewhere.

The quasiclassical $\overline{\partial}-$ dressing method is based on the nonlinear Beltrami equation for the function S in the auxuliary space of "spectral" parameter . This approach reveals a deep interrelation between the solutions of the eikonal equation (3.6) and the dVN hierarchy and the quasiconformal mappings on the plane [31]. One may suggest that the theory of the quasiclassical surfaces presented in the section 3 is closely related to the theory of quasiconformal mappings on the plane too.

Acknowledgements. This work is supported in part by COFIN PRIN " Sintesi" 2004.

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